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Comparison of several generating partitions of the Hénon map

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Abstract. The existence and uniqueness of generating partitions in non-hyperbolic dynamical systems is usually studied in a simple, exemplary system, namely the Hénon map at standard parameter values $a = 1.4$ and $b = 0.3$. We compare its standard partition with three other binary partitions, which are quite different from the standard partition but also appear to be generating. One of these partitions passes twice through the same orbit of homoclinic tangencies, providing a counterexample to a recent conjecture by Jaeger and Kantz (*J. Phys. A: Math. Gen.* **30** L567). Introducing some simple rules to manipulate symbolic sequences, we show how to translate symbolic sequences produced by one partition into sequences produced by the other partitions. This proves that all these partitions are as good approximations to generating partitions as the standard partition. We also construct an infinite number of binary partitions, which are all quite similar to the standard partition, derive their translation rules, and prove the same equivalence. It is not known for sure whether any of these partitions is indeed generating. But if one of them is generating, then they all are.

1. Introduction

Symbolic dynamics is a powerful tool to study chaotic dynamical systems (see e.g. [9, 11] for introductions). It is based on a partition of phase space into regions denoted by different symbols. For most purposes, the partition has to be 'generating', that is it must not assign the same symbolic sequence to different orbits. Unfortunately, one does not know how to construct these generating partitions in any but the most simple dynamical systems, namely one-dimensional maps and higher-dimensional systems that are uniformly hyperbolic. Even for the next simplest case, for two-dimensional dissipative maps, no systematic way is known to construct generating partitions.

The best construction rule we have is the one proposed in [7]: put the partition line through 'homoclinic tangencies', that is attractor points whose stable and unstable manifolds are parallel. This rule alone is, however, not sufficient to fix the partition line. Any homoclinic tangency has other homoclinic tangencies as images and preimages and thus defines a whole orbit of homoclinic tangencies, which can lie all over the attractor. Although the partition line should intersect each orbit of homoclinic tangencies at least once, one is still free to choose where the orbit is intersected. Most choices will not produce generating partitions. By choosing those homoclinic tangencies whose stable and unstable manifolds are only weakly curved one has successfully constructed partitions for two-dimensional dissipative maps [4, 7, 8], a three-dimensional dissipative flow [6], and a two-dimensional conservative map [1, 2]. The curvature of stable and unstable manifolds is, however, not invariant under coordinate transformations. Any construction scheme based on this curvature is therefore quite unsystematic.

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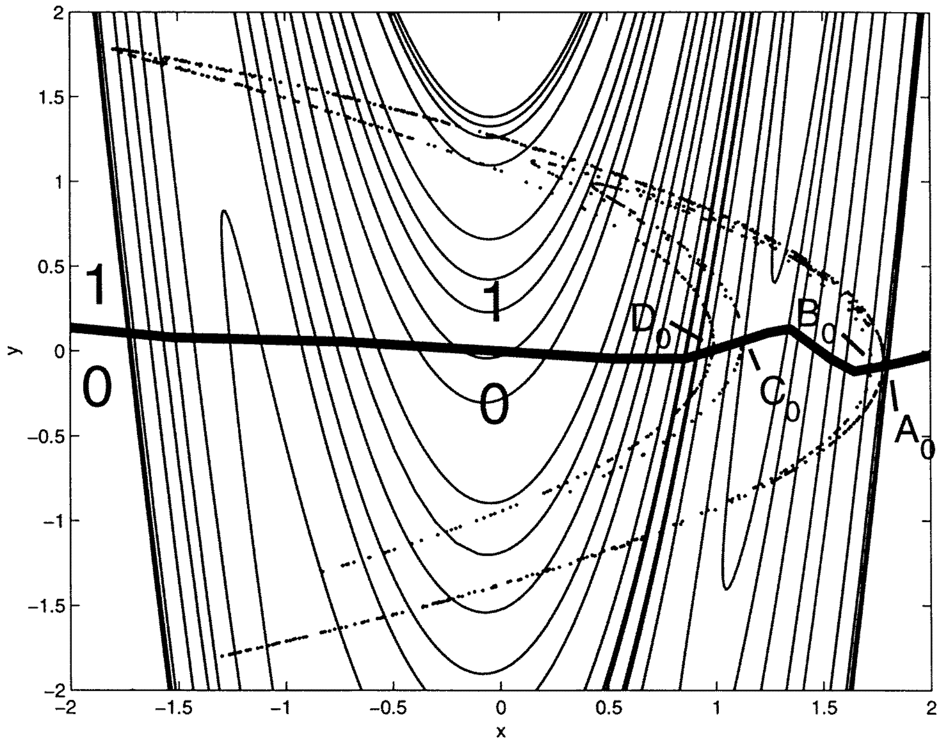

Figure 1. Hénon attractor at standard parameter values $a = 1.4$ and $b = 0.3$. The attractor is marked by dots. The thin solid line shows some segments of its stable manifold. The thick solid line denotes the approximate position of the standard partition line defined in [7]. It partitions the phase space into the two symbolic regions $\underline{0}$ and $\underline{1}$ and passes through homoclinic tangencies, that is points whose stable and unstable manifold are parallel. It was drawn here by connecting visually identified homoclinic tangencies. It crosses the attractor at the four places A_0, B_0, C_0, and D_0. Each of these crossings consists of an infinite number of homoclinic tangencies, which are too close to each other to be resolved.

Several studies have searched for more systematic construction rules [5, 8, 12]. Just like the original study [7], they focused on finding generating partitions for the two-dimensional, non-hyperbolic Hénon attractor [10], in the hope that knowing how to construct these partitions might help in constructing generating partitions for more complex dynamical systems. The most studied partition is the 01-partition at standard parameter values $a = 1.4$ and $b = 0.3$, which is shown in figure 1. Although there is no proof that the 01-partition is indeed generating, numerical tests have shown it to be at least a very good approximation to a generating partition [4, 7, 8, 12].

Is the 01-partition the only binary generating partition? For non-standard parameter values $a = 1$ and $b = 0.54$, two binary partitions were found a few years ago, which both seem to be generating [8]. Recently, we showed that even at standard parameter values, the 01-partition is not unique in being the only binary generating partition. There are other binary partitions, which are generating, if the 01-partition is generating. The same holds true for a wide range of parameters a and b, namely for all parameters where some short symbolic sequences are forbidden. We published these partitions in a technical report [5] and will summarize some of their properties here. Independently, Jaeger and Kantz discovered similar partitions [12]. All these partitions are shown in figure 2. Jaeger and Kantz also conjectured a necessary condition

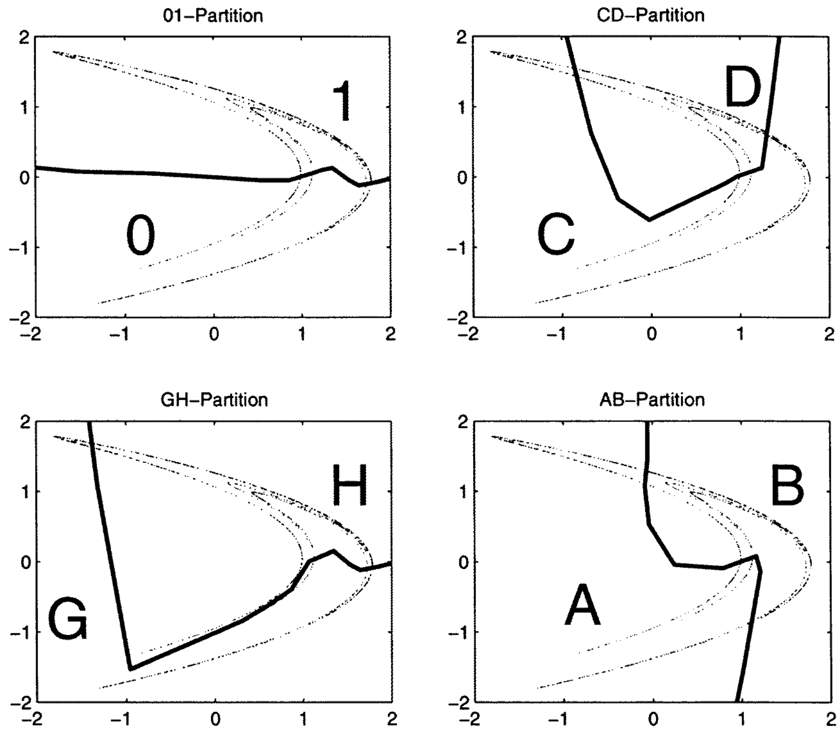

Figure 2. Four partitions of the Hénon attractor at standard parameter values. The 01-partition was the first to be discovered [7]. The CD-partition was found in [5, p 106], where its image was called the XY-partition, and independently in [12], where it was called partition VIII. The GH-partition was found in [12], where it was called partition IX. The AB-partition was found in [5, p 47]. These four partitions are related: if one of them is generating, then they all are.

for a partition to be generating: its partition line should intersect each orbit of homoclinic tangencies once, and only once [12, condition 1]. We will show here a counterexample to this conjecture: the AB-partition in figure 2 intersects some orbits of homoclinic tangencies twice.

In previous studies, like [4, 7, 8, 12], numerical tests were used to decide whether a partition is generating. Entropies were calculated and it was checked whether all periodic orbits of short length are classified correctly. Here we will proceed in a different, algebraic way, similar to the one used in [8] to compare two partitions of the Hénon attractor at non-standard parameter values. We will derive translation rules between the four partitions in figure 2. With these rules, one can translate any symbolic sequence from one partition into the corresponding symbolic sequences from the other partitions, without having to reconstruct the underlying orbit in phase space. In contrast to the two partitions at non-standard parameter values, for which no complete set of translation rules could be found in [8], the partitions presented here can be described with only a small number of translation rules. Using a suitable notation, these translation rules can be expressed concisely and applied easily.

We also will show that two different sequences never translate into the same sequence and conclude that the four partitions in figure 2 are 'equally good approximations to a generating partition'. That is, if one of them assigns different symbolic sequences to different orbits, then they all do. And if one of them is generating, then they all are. The same methods can be used to construct an infinite number of binary, generating partitions from the standard 01-partition, which are, however, almost indistinguishable from the 01-partition itself.

Section 2 reviews some basic definitions. Section 3 explains a simple way to calculate with symbolic sequences, which we will need for our translation rules. Section 4 describes the four partitions. Section 5 contains the translation rules and the proof that all four partitions are equally good approximations to a generating partition. Section 6 shows how to construct an infinite number of generating partitions from the 01-partition. Finally, section 7 discusses what these partitions teach us about the location of partition lines and the existence of symbol planes and which of the partitions are the simplest.

2. Basics

The Hénon map $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$ is given by

$$x_{n+1} = a - x_n^2 + b \cdot y_n \qquad y_{n+1} = x_n \tag{1}$$

with standard parameter values $a = 1.4$ and $b = 0.3$. Its attractor is shown by the dots in figure 1. Attractor points whose stable manifold lies parallel to their unstable manifold are called homoclinic tangencies. By construction [7], the 01-partition line intersects the attractor in some particular homoclinic tangencies, where neither the stable nor unstable manifold is too strongly curved. Attractor points on one side of the partition line are assigned the symbol $S_0 = 0$, points on the other side the symbol $S_0 = 1$. Points on the partition line can be included at will in either of the two symbolic regions.

Because of the fractal structure of the attractor, the shape of the 01-partition line may get quite complicated on a microscopic scale. As we will not try to prove that the 01-partition is generating, we will not have to discuss its exact position. Our arguments will hold for any partition line that is close to the one shown in figure 1.

As the Hénon map is invertible, each attractor point (x_0, y_0) defines an infinite orbit

$$\ldots (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots \tag{2}$$

of attractor points and thus an infinite sequence of symbols

$$\ldots S_{-2} S_{-1} \underline{S_0} S_1 S_2 \ldots \tag{3}$$

where S_n is the symbol of the region that contains (x_n, y_n). In such a sequence, we use the underscore '_' to mark the position of the present symbol S_0.

3. Manipulation of symbolic sequences

We will also work with symbolic sequences that are finite. By definition, such a sequence denotes the set of all attractor points whose infinite symbolic sequence contains this finite sequence at the correct place. Thus $\underline{0}$ denotes all the attractor points in the 0-region, $\underline{1}0$ denotes all the attractor points in the 1-region, whose image lies in the 0-region, and so on. We will use the symbol E (for 'everything') to fill in empty places in the middle of a symbolic sequence, if the corresponding symbolic region is not known. For example, $0\underline{E}E1$ denotes the set of all attractor points whose pre-image lies in the 0-region and whose second image lies in the 1-region. Applying the Hénon map f, we get $f(0\underline{E}E1) = 0E\underline{E}1$, the set of all attractor points whose second pre-image lies in the 0-region and whose image lies in the 1-region. In general, applying the map f to any symbolic sequence simply shifts the underscore '_' one place to the right.

As each finite symbolic sequence denotes a set of points, the usual operations of set algebra can be applied to them. For example:

$$\underline{0} \cup \underline{1} = \underline{E} = \text{ set of all attractor points} \tag{4}$$

$$\underline{0} \cap \underline{1} = \emptyset = \text{ empty set} \tag{5}$$

$$\underline{S} \cup \underline{E} = \underline{E} \qquad \text{for any symbol } S \tag{6}$$

$$\underline{S} \cap \underline{E} = \underline{S} \qquad \text{for any symbol } S \tag{7}$$

$$0\underline{E} \;\cap\; \underline{E}E1 \;=\; 0\underline{E}E1. \tag{8}$$

We will also use symbols other than 0, 1, or E in such expressions. Any symbol S can be used, if its symbolic region $\underline{S}$ is known.

4. Generating partitions of the Hénon attractor

A partition is called generating if it distinguishes all orbits, that is if it never assigns the same infinite symbolic sequence to two different orbits. Figure 2 shows some partitions which may be generating. They are all different from one another (and from each other's images and preimages) and therefore define four different symbolic dynamical systems. We will call them: 01-partition, CD-partition, GH-partition, and AB-partition.

The 01-partition can be used to define the other three partitions in figure 2. Their partition lines intersect the attractor in points that are images or preimages of the points through which the 01-partition line passes. Following the notation in [12], we divide the 01-partition line into four segments, denoted by A_0, B_0, C_0, and D_0 (see figure 1). The images of A_0 are called A_1, $A_2, \ldots,$ the preimages are called A_{-1}, $A_{-2}, \ldots.$ By definition, the CD-partition line passes first through A_{-2}, then C_0, D_0, and finally B_{-3}. The GH-partition line passes through A_0, B_0, C_0, and D_{-2}. And the AB-partition line passes through B_{-2}, C_0, D_0, B_{-1}, and A_{-1}. Note that the AB-partition line passes through both B_{-2} and B_{-1}, so that it intersects twice any orbit going through B_0.

5. Translation rules

To translate symbolic sequences from different partitions into one another, we first have to specify the contents of each symbolic region $\underline{A}$, $\underline{B}, \ldots.$ This can be done geometrically, by dividing the phase space into small regions corresponding to finite 01-sequences and by reassembling those regions to form the symbolic regions from figure 2. The result is

$$\underline{A} \;=\; \underline{E}0 \;\cup\; \underline{0}11 \tag{9}$$

$$\underline{B} \;=\; \underline{1}1 \;\cup\; \underline{0}10 \tag{10}$$

$$\underline{C} \;=\; \underline{0} \;\cup\; \underline{1}E0 \;\cup\; \underline{1}011 \tag{11}$$

$$\underline{D} \;=\; \underline{1}11 \;\cup\; \underline{1}010 \tag{12}$$

$$\underline{G} \;=\; 00\underline{0} \;\cup\; 1\underline{0} \;\cup\; \underline{1}00 \tag{13}$$

$$\underline{H} \;=\; 10\underline{0} \;\cup\; \underline{1}1 \;\cup\; \underline{1}01. \tag{14}$$

Instead of defining the partitions via their partition lines, we can regard equations (9)–(14) as the definitions of the AB-, CD-, and GH-partition in terms of the 01-partition. This approach is more general than the geometric definition of the partition line: it can be applied to any map, for which a binary 01-partition has been defined. Using the standard rules of set algebra, one can easily check that equations (9)–(14) define true partitions, that is the two symbolic regions are disjoint and their union covers the whole attractor. For example:

$$\underline{A} \cap \underline{B} = (\underline{E}0 \cup \underline{0}11) \cap (\underline{1}1 \cup \underline{0}10) \tag{15}$$
$$= (\underline{E}0 \cap \underline{1}1) \cup (\underline{E}0 \cap \underline{0}10) \cup (\underline{0}11 \cap \underline{1}1) \cup (\underline{0}11 \cap \underline{0}10) \tag{16}$$
$$= \emptyset \tag{17}$$
$$\underline{A} \cup \underline{B} = \underline{E}0 \cup \underline{0}11 \cup \underline{0}10 \cup \underline{1}1 \tag{18}$$
$$= \underline{E}0 \cup \underline{0}1 \cup \underline{1}1 = \underline{E}0 \cup \underline{E}1 \tag{19}$$
$$= \underline{E} \tag{20}$$

(and likewise $\underline{C} \cap \underline{D} = \underline{G} \cap \underline{H} = \emptyset$ and $\underline{C} \cup \underline{D} = \underline{G} \cup \underline{H} = \underline{E}$).

The rules (9) and (10) suffice to translate any infinite 01-sequence into a unique infinite AB-sequences. Given the 01-sequence, they determine whether the present symbolic region of the point is $\underline{A}$ or $\underline{B}$. By applying the map f one can shift these rules in time and determine the symbols A or B at all times, that is the whole infinite AB-sequence.

Thus any two orbits with the same 01-sequence have the same AB-sequence and any two orbits with different AB-sequences have different 01-sequences. We can express this by saying that the 01-partition is 'at least as good an approximation to a generating partition' as the AB-partition.

We want to show that the 01-partition and AB-partition are equally good approximations to a generating partition (by which we mean: two orbits are assigned different symbolic sequences by one partition if and only if they are assigned different symbolic sequences by the other partition). It remains to be shown that the AB-partition is at least as good an approximation to a generating partition as the 01-partition. For this, all we need are rules that translate infinite AB-sequences back into 01-sequences. It is not immediately obvious whether such rules can be found. Their existence depends on the grammatical rules of the 01-symbolic dynamics, that is on which 01-sequences are forbidden. For example, both the sequence $\ldots 00\underline{0}100 \ldots$ and the sequence $\ldots 00\underline{1}100 \ldots$ translate into the sequence $\ldots AA\underline{B}AAA \ldots$. If both 01-sequences were allowed, then there would be no unique way to translate the AB-sequence back into a 01-sequence.

For the Hénon attractor at standard parameter values, the first few grammatical rules of the 01-symbolic dynamics are well known [8]. The shortest forbidden sequences are 0000, 0010, and 0110. These are all the grammatical rules one needs to translate AB-, CD-, and GH-sequences back into 01-sequences.

As an example, let us derive the translation rules of AB-sequences into 01-sequences. Using the definitions (9) and (10), we get

$$A\underline{A} = (\underline{0} \cup 0\underline{1}1) \cap (\underline{E}0 \cup \underline{0}11) = \underline{0}0 \cup \underline{0}11 \tag{21}$$
$$A\underline{A}B = (\underline{0}0 \cup \underline{0}11) \cap (\underline{E}11 \cup \underline{E}010) = \underline{0}010 \cup \underline{0}11. \tag{22}$$

But $\underline{0}010 = \emptyset$, so that we can already translate $\underline{0}11$ into $A\underline{A}B$. As the sets $\underline{E}0$ and $\underline{0}11$ are disjoint, we get from equation (9):

$$\underline{E}0 = \underline{A} \setminus \underline{0}11 = \underline{A} \setminus A\underline{A}B = B\underline{A}B \cup \underline{A}A \tag{23}$$
$$\underline{0} = BA\underline{B} \cup A\underline{A} \tag{24}$$
$$\underline{1} = \underline{E} \setminus \underline{0} = AA\underline{B} \cup B\underline{E}. \tag{25}$$

(As usual, $M \setminus N$ denotes the set of all points that are in M but not in N.) The last two equations (24) and (25) are the desired translation rules. With them, any infinite AB-sequence can be translated into a unique 01-sequence. To conclude, if 0010 is a forbidden sequence, then the 01-partition and the AB-partition are equally good approximations to a generating partition.

Because of the other forbidden 01-sequences, these back-translation rules can be expressed in slightly other forms. If 0110 is forbidden, we can derive the equation $\underline{0}11 = A\underline{A}BB$ and use it to get the alternative rules:

$$\underline{0} = BA\underline{B} \cup A\underline{A} \cup AA\underline{B}A \tag{26}$$

$$\underline{1} = AA\underline{B}B \cup B\underline{E} \tag{27}$$

which hold even if 0010 is not forbidden. At standard parameter values, where both 0010 and 0110 are forbidden, either pair of rules can be used to translate AB-sequences back into 01-sequences.

In an analogous way, rules can be found to translate infinite CD-sequences and GH-sequences back into 01-sequences. A simple derivation uses the equations $CC\underline{D} = 0\underline{1}1$, $D\underline{C}DD = \emptyset$, and $\underline{D} \cup C\underline{C}D \cup \underline{C}CD = \underline{E}E1$. Let us just state the results. CD-sequences can be translated by

$$\underline{0} = DCD\underline{C} \cup CC\underline{C} \tag{28}$$

$$\underline{1} = \underline{D} \cup DE\underline{C} \cup CCD\underline{C} \tag{29}$$

if the sequences 0010 and 0110 are forbidden. GH-sequences can be translated by

$$\underline{0} = \underline{G}H \cup \underline{G}GG \cup GG\underline{H} \tag{30}$$

$$\underline{1} = H\underline{H} \cup HG\underline{H} \cup \underline{G}GH \tag{31}$$

if the sequence 0000 is forbidden. (A simple derivation uses $G\underline{G}H = 1\underline{0}0$ and $\underline{G} = (\underline{0} \setminus 10\underline{0}) \cup \underline{1}00$.)

For the Hénon attractor at standard parameter values, the sequences 0000, 0010 and 0110 are all forbidden, and we can conclude that the 01-, AB-, CD- and GH-partition are equally good approximations to a generating partition. If one of them assigns different symbolic sequences to two different orbits, then they all do. And if one of them is a generating partition, then they all are.

6. An infinite number of generating partitions

The three partitions in figure 2 are not the only generating partitions that can be constructed from the 01-partition, if it is generating. They are special in that they differ quite strongly from the 01-partition, yet appear geometrically simple. We will now show how to construct other binary generating partitions, which are quite similar to the 01-partition, by taking a small part from the 1-region and adding it to the 0-region.

To construct such a partition, one first has to choose a finite sequence, which we will call the 'difference'-sequence. It should be allowed, but should be turned, by changing only the underlined symbol, into a forbidden sequence. As an example, we choose the difference-sequence $00\underline{1}110111$, which is allowed (see figure 3), while the sequence $00\underline{0}110111$ is forbidden (because it contains the forbidden sequence 0110).

Next, one takes the small region $00\underline{1}110111$ from the 1-region and adds it to the 0-region. We will describe the new partition with the symbols $\acute{0}$ and $\acute{1}$:

$$\underline{\acute{0}} = \underline{0} \cup 00\underline{1}110111 \tag{32}$$

$$\underline{\acute{1}} = \underline{1} \setminus 00\underline{1}110111. \tag{33}$$

The $\acute{0}\acute{1}$-partition is quite similar to the original 01-partition. If we translate a typical 01-sequence, most symbols 1 will be translated into the symbol $\acute{1}$ and most symbols $\acute{0}$ will be generated by translating the symbol 0. In contrast to the forbidden sequence $00\underline{0}110111$, its

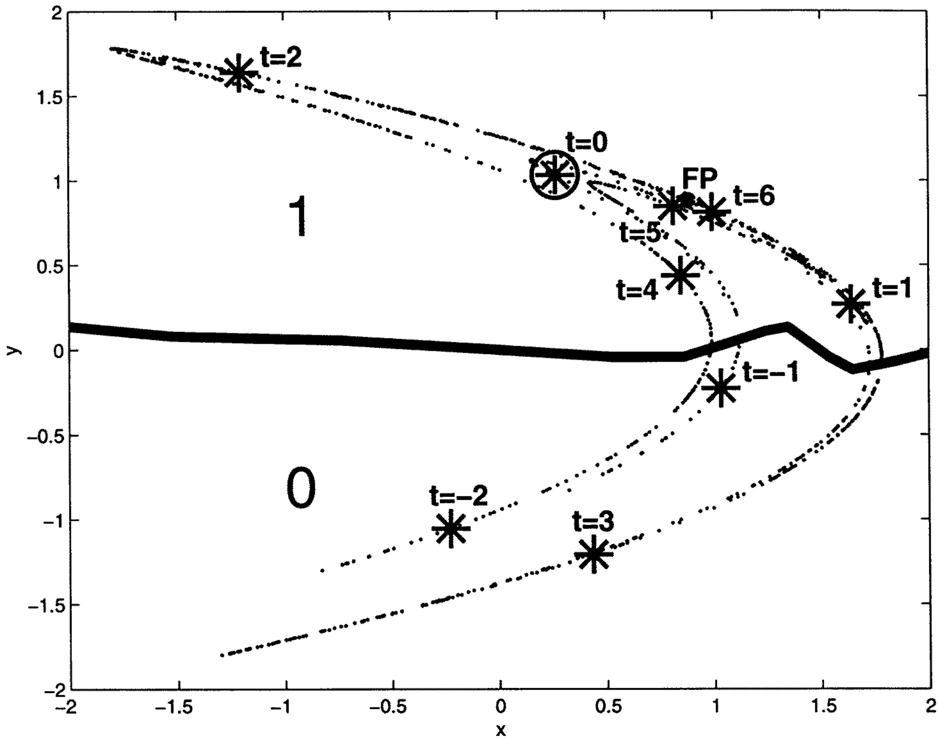

Figure 3. More generating partitions for the Hénon-attractor at standard parameter values. An orbit (stars) is shown at times $t = -2$ to 6, which generates the symbolic sequence $\ldots 00\underline{1}110111\ldots$. At times $t = 5$ and $t = 6$ it passes near the unstable fixpoint (FP, marked by large dot). At time $t = 0$ it lies in a small region (smaller than the circle) that can be added to the 0-region, while still keeping the partition generating. By choosing smaller and smaller regions, an infinite number of binary partitions can be constructed, which are as good approximations to a generating partition as the standard 01-partition (see text for proof).

counterpart $\acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1}$ may be allowed; and a likely way in which it can be generated is by translating the difference-sequence $00\underline{1}110111$. If we can show that both sequence describe the same set of attractor points, then we can reverse the translation rules (32) and (33) and prove that the $\acute{0}\acute{1}$-partition is generating, too. The difference-sequence $00\underline{1}110111$ was chosen to allow such a proof.

To show that $\acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1} = 00\underline{1}110111$, one uses the definition (32) to derive conditions under which $\acute{0}$-symbols are translated into 0-symbols. One such condition is $\underline{\acute{0}}EE\acute{1} = \underline{0}EE\acute{1}$ (as $\underline{E}EE\acute{1} \cap 00\underline{1}110111 = 0$), and it alone suffices to translate all the symbols $\acute{0}$ in our sequence, except the underlined one:

$$\acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1} = 00\underline{\acute{0}}\acute{1}\acute{1}0\acute{1}\acute{1}\acute{1}. \tag{34}$$

In the same way, one can find conditions under which the symbol $\acute{1}$ translates into 1. The conditions $\acute{1}\underline{\acute{1}} = \acute{1}\underline{1}$, $1E\underline{\acute{1}} = 1E\underline{1}$, and $\underline{\acute{1}}E0 = \underline{1}E0$ suffice to show that:

$$00\underline{\acute{0}}\acute{1}\acute{1}0\acute{1}\acute{1}\acute{1} = 00\underline{\acute{0}}110111. \tag{35}$$

And, because $00\underline{0}110111$ is forbidden, this last expression is equal to $00\underline{1}110111$. Putting everything together, we get

$$\acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1} = 00\underline{1}110111 \tag{36}$$

$$\underline{0} = \underline{\acute{0}} \setminus \acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1} \tag{37}$$

$$\underline{1} = \underline{\acute{1}} \cup \acute{0}\acute{0}\underline{\acute{0}}\acute{1}\acute{1}\acute{0}\acute{1}\acute{1}\acute{1}\acute{1}. \tag{38}$$

These back-translation rules prove that the $\acute{0}\acute{1}$-partition is as good an approximation to a generating partition as the 01-partition.

Similar partitions can be constructed starting from other forbidden sequences and the corresponding difference-sequences. The resulting partition will not always be generating, especially if the difference-sequence is short. If we had started, for example, from the forbidden sequence $\underline{0}110$, the difference sequence $\underline{1}110$, and the new partition $\underline{\tilde{0}} = \underline{0} \cup \underline{1}110$ and $\underline{\tilde{1}} = \underline{1} \setminus \underline{1}110$, we would not have been able to show that $\underline{1}110 = \underline{\tilde{0}}\tilde{1}\tilde{1}\tilde{0}$. This is because the 01-partition and the $\tilde{0}\tilde{1}$-partition are too different and the symbol $\tilde{0}$ at the right end of $\underline{\tilde{0}}\tilde{1}\tilde{1}\tilde{0}$ may correspond not to a 0, but to a 1, like in $\underline{0}111110 \subseteq \underline{\tilde{0}}\tilde{1}\tilde{1}\tilde{0}$.

If one starts, however, with sufficiently long difference-sequences, such problems are less common, and back-translation rules can often be derived. (This is because the number of symbols $\acute{0}$ or $\tilde{0}$ that have to be translated into 0 during the derivation of the back-translation rule grows only linearly with the length of the difference-sequence, while the chance that any of them does not correspond to 0 decays exponentially with the number of conditions we can derive to translate it into 0, that is exponentially with the length of the difference-sequence.) In particular, one can easily show, by repeating the above calculation, that adding more symbols 1 to the right of the difference-sequence $00\underline{1}110111$ causes no problem with the back-translation and that any partition with the symbolic regions

$$\underline{0} \cup 00\underline{1}110111\ldots1 \quad \text{and} \tag{39}$$

$$\underline{1} \setminus 00\underline{1}110111\ldots1 \tag{40}$$

is generating if the 01-partition is. Figure 3 shows that all these difference-sequences $00\underline{1}110111\ldots1$ are allowed, no matter how long they are (as the orbit shown passes the vicinity of the unstable fixpoint, where arbitrarily long sequences $111\ldots1$ can be generated). Thus we can construct an infinite number of different generating partitions. which are, however, all quite similar to the original 01-partition.

Although this construction scheme is general enough to be applicable to other partitions and dynamical systems, it does not produce all generating partitions. To construct all of them, including the three other partitions in figure 2, one would have to find a more powerful construction scheme.

7. Discussion

7.1. Symmetries in the translation rules

Is there any systematic way to construct all generating partitions? We do not know, but there are some curious symmetries between the translation rules (9)–(14) and the back-translation rules (24)–(31), which give a hint that such a system might exist.

For example, the translation rules (9) and (10) are turned into the back-translation rules (24) and (25) by interchanging 0 and B, 1 and A, and future and past (that is left and right) in all symbolic sequences. As shown above, the translation rules (24) and (25) depend on the sequence 0010 being forbidden. Under the symmetry, this sequence turns into $BABB$, which is also forbidden. But the symmetry does not generalize to other grammatical rules, which are irrelevant for the translation rules. For example, the counterparts of the forbidden sequences 0110, 0000 and $AABA$ are the allowed sequences $BAAB$, $BBBB$ and 1011.

Similarly, the rules (11) and (12) are turned into the rules (28) and (29) by interchanging 0 and D, 1 and C, and future and past in all symbolic sequences. The forbidden sequences

0010 and 0110 are thereby turned into the sequences $DCDD$ and $DCCD$, which are also forbidden. And finally, one can turn the rules (13) and (14) into the rules (30) and (31) by interchanging 0 and G, 1 and H, and future and past in all sequences. The forbidden sequence 0000 is thereby turned into the sequence $GGGG$, which is also forbidden.

All these symmetries are not at all obvious from the geometric relationships in figure 2. Whether they indicate a hidden systems, or whether they are mere coincidences, remains unknown. In some other translation rules, like those between AB-sequences and CD-sequences, an analogous symmetry could not be found. The similarity between the translation rules (32) and (33) and the back-translation rules (37) and (38) is simply due to the way we constructed the $\acute{0}\acute{1}$-partition; it is not an exact symmetry.

7.2. Existence of symbol planes

Given this large variety of binary partitions, how can we decide which one is the most simple? One way is to check which partition defines the most simple symbol plane. Symbol planes were introduced in [3]. They are constructed from the original, two-dimensional phase space by applying a discontinuous coordinate transformation, which fulfils these conditions: All point sets described by semi-infinite future sequences $\underline{S_0}S_1S_2\ldots$ are transformed into vertical lines in the symbol plane. All point sets described by semi-infinite past sequences $\ldots S_{-2}S_{-1}\underline{E}$ are transformed into horizontal lines. And while distances may change, the order of attractor points along these lines is maintained. Due to these conditions, the temporal dynamics in the symbol plane is simply described by a piecewise linear map. It stretches all horizontal lines, cuts them at the partition line, and folds them back on top of each other. In this way, the intuitive notion that the map stretches and folds its attractor can be made mathematically precise.

The symbol plane for the 01-partition of the Hénon attractor was constructed in [3]. Can analogous symbol planes be constructed with the other partitions, that is can their sequences be ordered vertically and horizontally in a consistent way? They can for the CD-partition. Figures 4(*a*) and (*b*) show schematically, how CD-sequences can be ordered horizontally and vertically. The action of the Hénon-map on the CD-symbol plane is just as simple as its action on the 01-symbol plane (results not shown), although the resulting picture of how the attractor is stretched and folded is quite different.

But no symbol plane can be constructed for the AB-partition or the GH-partition. The problem with ordering AB-sequences is illustrated in figures 4(*c*) and (*d*). Point sets of the form $\underline{A}AB\ldots$, $\underline{B}BA\ldots$, $\ldots BAA\underline{E}$, and $\ldots BBB\underline{E}$ (where '...' stands for some allowed semi-infinite AB-sequences) intersect in four corners and thereby form a quadrangle with curved sides. Note that one of its corners is directed inward. In the case of the 01- or CD-partition, all such quadrangles have corners that are directed outwards. They can therefore be transformed into rectangles with vertical and horizontal sides in the symbol plane. But as figure 4(*d*) shows, quadrangles with an inwardly directed corner cannot be transformed into rectangles in the symbol plane without violating the order of attractor points along the sides of the quadrangle. Thus no symbol plane can be constructed for the AB-partition, at least none where symbolic sequences with the same future part $\underline{S_0}S_1S_2\ldots$ lie on vertical lines and sequences with the same past part $\ldots S_{-2}S_{-1}\underline{E}$ lie on horizontal lines. The best one could probably achieve is to construct symbol planes for the points in $\underline{A}$ and $\underline{B}$ separately, so that $\ldots S_{-2}S_{-1}\underline{E}$ consists of two horizontal lines, one in each symbol plane. This would describe yet another, much more complicated way of how the Hénon-map stretches and folds its attractor.

Constructing a symbol plane for the GH-partition or any of the partitions in section 6

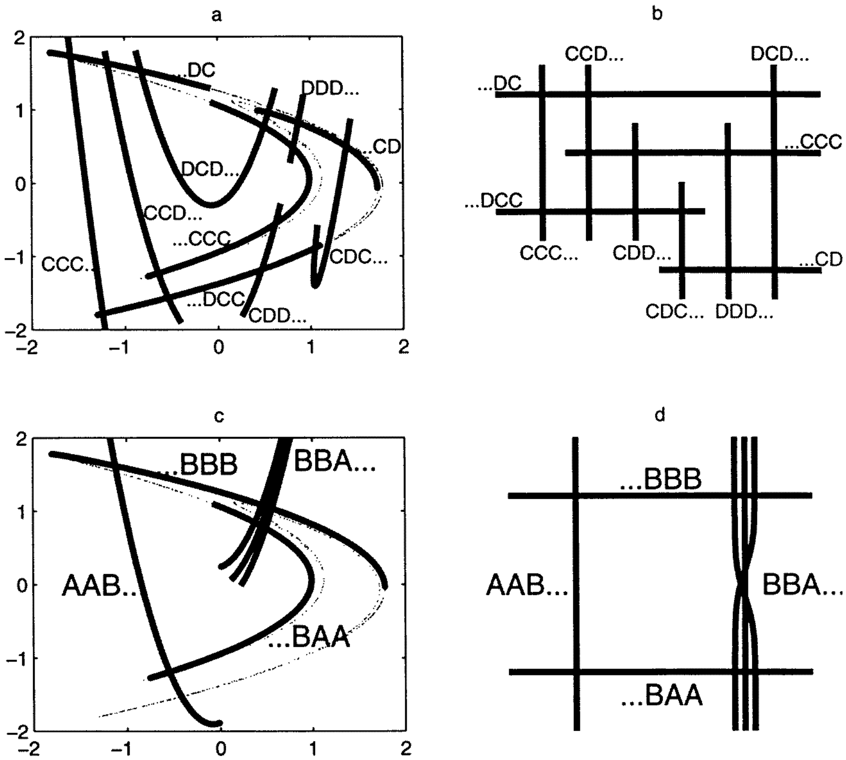

Figure 4. CD-sequences can be ordered in a symbol plane, but AB-sequences cannot. (*a*) shows as an example the point sets $\ldots CD\underline{E}$, $\ldots DCC\underline{E}$, $\ldots CCC\underline{E}$, and $\ldots DC\underline{E}$ (with '...' standing for some semi-infinite CD-sequences), which are segments of the unstable manifold. If one constructs a symbol plane, these segments have to be transformed into horizontal lines. Also shown are segments of the stable manifold, on which the point sets $\underline{C}CC\ldots$, $\underline{C}CD\ldots$, $\underline{C}DD\ldots$, $\underline{C}DC\ldots$, $\underline{D}DD\ldots$ and $\underline{D}CD\ldots$ lie. They have to be transformed into vertical lines in a symbol plane. (*b*) shows schematically, how such a vertical and horizontal ordering can be achieved. (*c*) shows similar point sets for the AB-partition: $\ldots BBB\underline{E}$ and $\ldots BAA\underline{E}$ have to be mapped onto horizontal lines and $\underline{A}AB\ldots$ and the three segments starting with $\underline{B}BA\ldots$ have to be mapped onto vertical lines in the symbol plane (The last three segments differ in the '...' part of their sequence: they are $\underline{B}BABAAA\ldots$, $\underline{B}BABAAB\ldots$, and $\underline{B}BABABA\ldots$). If one tries to order all those segments vertically and horizontally, one always ends up with an inconsistency, like the one in (*d*), where the three segments of the form $\underline{B}BA\ldots$ cannot be vertical lines, because they cross each other.

leads to similar inconsistencies (results not shown). Thus, even partitions that are equally good approximations to generating partitions can be further distinguished on the ground of whether their symbol sequences can be ordered in a symbol plane or not.

7.3. Location of partition lines

Jaeger and Kantz [12] conjectured that any (binary) generating partition should pass through each orbit of homoclinic tangencies only once. This conjecture was based on the observation that the 01-, CD-, and GH-partition, which were assumed to be generating, intersect orbits through the segments A_0, B_0, C_0, and D_0 only once. But the AB-partition, which we presented in figure 2, intersects orbits through the segment B_0 twice, namely at B_{-1} and B_{-2}. And we could prove that the AB-partition is generating, if the 01-partition is generating. The AB-

partition (and also the partitions in section 6) therefore provide counterexamples to the above conjecture.

But perhaps the same condition can be used as a criterion of simplicity. Indeed, the only two known partitions that meet this condition, as well as the other geometric condition discussed in [12], are the 01-partition and the CD-partition, that is the same partitions for which the most simple symbol planes can be constructed.

It will, however, be hard to generalize these geometric criteria of simplicity to higher-dimensional systems. As the construction of symbol planes depends on the order of points along the one-dimensional stable and unstable manifolds, it will be difficult to construct symbol spaces for higher-dimensional manifolds, where there is no such order. Likewise, although cutting the attractor at homoclinic tangencies may be enough to construct generating partitions in two-dimensional systems, it may not be enough in higher-dimensional systems, where stable and unstable manifolds can fold in many more ways. How to construct generating partitions in these higher-dimensional systems, and how to decide which of them is the simplest, remain open questions.

8. Conclusion

Although we cannot prove that any of the partitions in figure 2 or any of those in section 6 are indeed generating, it was easy to prove that they are equally good approximations to a generating partition: if two orbits get assigned different symbolic sequences by one partition, then they get assigned different symbolic sequences by all of these partitions. This holds not only for standard parameters of the Hénon map, but for all parameter values where some short 01-sequences are forbidden.

Neither the known geometric [12] nor algebraic [5] methods provide a systematic way to construct all generating partitions for a given attractor. But the simplicity of the construction scheme in section 6, which produced an infinite number of generating partitions for the Hénon attractor, suggests that a large variety of generating partitions exists, and that constructing all of them will be very difficult. The same will probably hold for other dynamical systems with forbidden symbolic sequences, including one-dimensional maps.

Which of these partitions is the most simple? According to different criteria, the CD-partition and the 01-partition are the most simple among the known partitions of the Hénon attractor. But there seems to be no commonly accepted criterion by which one of them can be shown to be more simple than the other one. The 01-partition agrees well with the intuitive idea of folding and stretching, that underlies the construction of the Hénon attractor [10]. The CD-partition provides a different, but equally simple picture of how the Hénon attractor is stretched and folded. To conclude, even in the case of the rather simple Hénon attractor, an infinite variety of generating partitions exist, and it is not clear which one is the most simple.

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